Induced Metric And Matrix Inequalities On Unitary Matrices

H. F. Chau,[1, *] Chi-Kwong Li,[2, †] Yiu-Tung Poon,[3, ‡] and Nung-Sing Sze[4, §]

[1] *Department of Physics and Center of Theoretical and Computational Physics, University of Hong Kong, Pokfulam Road, Hong Kong*

[2] *Department of Mathematics, College of William & Mary, Williamsburg, VA 23187-8795, USA*[¶]

[3] *Department of Mathematics, Iowa State University, Ames, IA 50011, USA*

[4] *Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong*

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Recently, Chau [Quant. Inform. & Comp. **11**, 721 (2011)] showed that one can define certain metrics and pseudo-metrics on $U(n)$, the group of all $n \times n$ unitary matrices, based on the arguments of the eigenvalues of the unitary matrices. More importantly, these metrics and pseudo-metrics have quantum information theoretical meanings. So it is instructive to study this kind of metrics and pseudo-metrics on $U(n)$. Here we show that any symmetric norm on $\mathbb{R}^n$ induces a metric on $U(n)$. Furthermore, using the same technique, we prove an inequality concerning the eigenvalues of a product of two unitary matrices which generalizes a few inequalities obtained earlier by Chau [arXiv:1006.3614v1].

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I. INTRODUCTION

In quantum information science, it is instructive to measure the cost needed to evolve a quantum system [1] as well as to quantify the difference between two quantum evolutions on a system [2]. To some extent, the solutions of both problems are closely related to certain pseudo-metric functions on unitary operators. To see this, suppose we are given a certain quantifiable cost required to implement a unitary operation acting on an n-dimensional Hilbert space. We may represent this cost by a non-negative function $f\colon U(n) \to \mathbb{R}$, where $U(n)$ is the group of all $n \times n$ unitary matrices. The larger the value of $f(X)$, the higher the cost of implementing the unitary operation X. Besides, $f(X) = 0$ if it is costless to perform X. We may further require this cost function f to satisfy the following constraints.

Constraints for the cost function f:

1. $f(e^{ir}X) = f(X)$ for all $r \in \mathbb{R}$ and $X \in U(n)$. In addition, $f(I) = 0$. The underlying reason is that changing the global phase of X has no effect on the quantum system. Besides, the identity operation does not change any quantum state and hence should be costless.

2. $f(X^{-1}) = f(X)$ for all $X \in U(n)$. This is because X^{-1} can be implemented by running the quantum circuit for X backward in time with the same cost.

3. $f(Y^{-1}XY) = f(X)$ for all $X, Y \in U(n)$. The rationale is that the cost to evolve a quantum system should be eigenbasis independent. Although this assumption is questionable for bipartite systems, we will stick to it in this paper for the evolution cost for monopartite system is already a worthy topics to investigate.

4. $f(XY) \leq f(X) + f(Y)$ for all $X, Y \in U(n)$. The reason behind is that a possible way to implement XY is to first apply Y then follow by X. If we further demand that the cost is additive (in the sense that the cost of applying Y and then X is equal to the cost of applying Y plus the cost of applying X), which is not an unreasonable demand after all, then the inequality follows.

A cost function f induces a function $d\colon U(n) \times U(n) \to \mathbb{R}$ by the equation $d(X,Y) = f(XY^{-1})$ for all $X, Y \in U(n)$. Surely, $d(X,Y)$ can be regarded as the cost needed to transform Y to X. In this respect, the induced function d provides a partial answer to the problem of quantifying the difference between two quantum evolutions on a system. The larger the value of $d(X,Y)$, the more different the quantum operations X and Y is. More importantly, since f obeys the above four constraints, $d(\cdot,\cdot)$ must be pseudo-metric on $U(n)$ because it satisfies $d(X,Y) \geq 0$, $d(X,Y) = d(Y,X)$ and $d(X,Z) \leq d(X,Y) + d(Y,Z)$ for all $X, Y, Z \in U(n)$. Nevertheless, $d(\cdot,\cdot)$ is not a metric for $d(X,Y) = 0$ does not imply $X = Y$. We remark that the induced d also obeys $d(ZX,ZY) = d(X,Y)$ for all $X, Y, Z \in U(n)$.

Conversely, suppose there is a pseudo-metric d on $U(n)$ quantifying the difference between two unitary operations acting on a n-dimensional quantum system. Surely, it should satisfy $d(X,X) = 0$, $d(e^{ir}X,Y) = d(X,Y)$ and $d(X,Y) = d(ZX,ZY)$ for all $r \in \mathbb{R}$, $X, Y, Z \in U(n)$. The reason is that the difficulty in distinguishing between two unitary operations is unchanged by varying the global

* Corresponding author, hfchau@hku.hk
† ckli@math.wm.edu
‡ ytpoon@iastate.edu
§ raymond.sze@inet.polyu.edu.hk
¶ (in the spring of 2012) Department of Mathematics, University of Hong Kong, Pokfulam Road, Hong Kong

phase in one of the operations and by applying a common quantum operation to them. (Again, this reason is valid as we restrict our study to monopartite systems.) More importantly, d induces the function $f(X) = d(X, I)$ on $U(n)$ which obeys the four constraints on f. (The second constraint follows from $f(X^{-1}) = d(X^{-1}, I) = d(X\,X^{-1}, X) = d(I, X) = d(X, I) = f(X)$. And the other three constraints can be proven in a similar way.) To summarize, we have argued that the cost function f describing the resources required to evolve a (monopartite) quantum system is equivalent to quantifying the difference between two quantum evolutions on a (monopartite) system through the induced pseudo-metric function d. And we remark on passing that our discussions so far are valid for infinite-dimensional quantum systems as well.

Recently, Chau [3, 4] introduced a family of cost functions on $U(n)$ based on a tight quantum speed limit lower bound on the evolution time of a quantum system he discovered earlier [5]. In quantum information science, these cost functions can be interpreted as the least amount of resources (measured in terms of the product of the evolution time and the average absolute deviation from the median of the energy) needed to perform a unitary operation $X \in U(n)$ [4]. With the above quantum information science meaning in mind, it is not surprising that each cost function in this family depends only on the eigenvalues of its input argument X. Actually, it can be written as a certain weighted sum of the absolute value of the argument of the eigenvalues of X [3, 4].

By eigenvalue perturbation method, Chau [3, 4] proved that for each cost function in the family, the corresponding induced function d is indeed a pseudo-metrics on $U(n)$ (and hence the cost function really satisfies the four constraints listed earlier). In fact, he proved something more. In addition to this induced family of pseudo-metrics, he also discovered a family of closely related metrics on $U(n)$. The only difference between them is that the family of metrics is an "un-optimized" version of the family of metrics in the sense that it does not take into account the fact that altering the global phase of a unitary operation does not affect the cost at all [3, 4]. More precisely, the underlying cost functions for the family of metrics obey the four constraints listed above except that the first one is replaced by $f(X) = 0$ if and only if $X = I$. Note that given $X, Y \in U(n)$, the family of metrics can also be expressed as certain weighted sums of the absolute value of the argument of the eigenvalues of the matrix XY^{-1} [3, 4].

Interestingly, the family of metrics on $U(n)$ discovered by Chau provides another partial answer to the problem of quantifying the difference between two quantum evolutions on a system. Specifically, Chau [3, 4] showed that the metric functions he discovered can be used to give a quantitative measure on the degree of non-commutativity between two unitary matrices X and Y in terms of certain resources needed to transform XY to YX.

The above background information shows that a number of quantum information science questions are related to the cost function f (or equivalently, the pseudo-metric or its "un-optimize" metric version d). Besides, the third constraint for f, namely, $f(Y^{-1}XY) = f(X)$ for all $X, Y \in U(n)$, implies that the cost function f depends on the eigenvalues of its input argument only. Equivalently, it means that the corresponding metric and pseudo-metric $d(X, Y)$'s on $U(n)$ are functions of the eigenvalues of XY^{-1} only.

In this paper, we adopt the following strategy to investigate the problem of metrics, pseudo-metrics and their relation with quantum information science. We begin by finding metrics and pseudo-metrics $d(X, Y)$'s on $U(n)$ that are functions of the eigenvalues of XY^{-1} only by means of Proposition 2. More precisely, we prove that a symmetric norm of $\mathbb{R}^n$ induces a metric and a pseudo-metric on $U(n)$ of the required type. We then show in Example 4 that some of the new metrics and pseudo-metrics discovered in this way indeed have quantum information science meanings. Interestingly, Proposition 2 has merit on its own for we can adapt its proof to show an inequality concerning the eigenvalues of a product of two unitary matrices. This inequality is a generalization of several inequalities first proven in Ref. [3] using eigenvalue perturbation technique. Finally, we briefly discuss the connection of our findings and the Horn's problem on eigenvalue inequalities for the sum of Hermitian matrices.

II. METRIC AND PSEUDO-METRIC INDUCED BY A SYMMETRIC NORM

To show that a symmetric norm on $\mathbb{R}^n$ induces a metric and a pseudo-metric on $U(n)$, we make use of the following result by Thompson [6]:

Theorem 1 (Thompson)**.** If A and B are Hermitian matrices, there exist unitary matrices X and Y (depending on A and B) such that

$$\exp(iA)\exp(iB) = \exp\left(iXAX^{-1} + iYBY^{-1}\right). \quad (1)$$

Note that Thompson proved his result by assuming the validity of the Horn's conjecture concerning the relation of the eigenvalues of the Hermitian matrices A, B, and $C = A + B$. The Horn's conjecture was confirmed based on the works of Klyachko [7] and Knutson and Tao [8]; see Ref. [9] for an excellent survey of the results. Later, Agnihotri and Woodward [10] improved the result of Thompson and gave a necessary and sufficient condition for the eigenvalues of (special) unitary matrices X, Y and $Z = XY$ using quantum Schubert calculus. The proof is technical and the statement of the result involve a large set of inequalities on the arguments of the eigenvalues of the unitary matrices X, Y and $Z = XY$ by putting them in suitable interval $[r, r + 2\pi)$. So, it is not easy to use. In fact, it suffices (and is actually more

practical) to use Theorem 1 to derive our results. We will further discuss the connection between our results with the Horn's problem in Section IV. We first present our results in the following.

Recall that a symmetric norm $g\colon \mathbb{R}^n \to [0,\infty)$ is a norm function such that $g(\mathbf{v}) = g(\mathbf{v}P)$ for any $\mathbf{v} \in \mathbb{R}^{1\times n}$, and any permutation matrix or diagonal orthogonal matrix P.

Proposition 2. Let $g : \mathbb{R}^n \to [0,\infty)$ be a symmetric norm. We may define a metric on $U(n)$ as follows:

$$d_g(X,Y) = g(|a_1|,\dots,|a_n|), \tag{2}$$

where XY^{-1} has eigenvalues e^{ia_j}'s with $\pi \geq a_1 \geq \cdots \geq a_n > -\pi$. Furthermore, we may define a pseudo-metric on $U(n)$ by

$$d_g^{\triangledown}(X,Y) = \inf_{r\in\mathbb{R}} g(|a_1(r)|,\dots,|a_n(r)|), \tag{3}$$

where $e^{ir}XY^{-1}$ has eigenvalues $e^{ia_j(r)}$'s with $\pi \geq a_1(r) \geq \cdots \geq a_n(r) > -\pi$.

Note that the infimum above is actually a minimum as we can search the infimum in any compact interval of the form $[r_0, r_0 + 2\pi]$.

Proof. Surely $d_g(X,Y)$, $d_g^{\triangledown}(X,Y) \geq 0$ for all $X,Y \in U(n)$. Besides, $d_g(X,X) = g(0,0,\dots,0) = 0$. And if $X \neq Y$, at least one eigenvalue of XY^{-1} must be different from 1. Since g is a norm, we conclude that $d_g(X,Y) > 0$.

Suppose XY^{-1} has eigenvalues e^{ia_j}'s with $\pi \geq a_1 \geq \cdots \geq a_n > -\pi$. Clearly, the eigenvalues of YX^{-1} are e^{-ia_j}'s. As g is a symmetric norm, $g(|a_1|,\dots,|a_n|) = g(|-a_n|,\dots,|-a_1|)$. Hence, $d_g(X,Y) = d_g(Y,X)$. By applying the same argument to $e^{ir}XY^{-1}$, we get $d_g(e^{ir}X,Y) = d_g(Y,e^{ir}X)$ for all $r \in \mathbb{R}$. From Eqs. (2) and (3), we know that $d_g^{\triangledown}(X,Y) = \inf_{r\in\mathbb{R}} d_g(e^{ir}X,Y) = \inf_{r\in\mathbb{R}} d_g(X,e^{-ir}Y)$. Hence, $d_g^{\triangledown}(X,Y) = d_g^{\triangledown}(Y,X)$.

Finally, we verify the triangle inequalities for $d_g(\cdot,\cdot)$ and $d_g^{\triangledown}(\cdot,\cdot)$. Let $X,Y,Z \in U(n)$. Suppose $d_g(X,Y) = g(|a_1|,\dots,|a_n|)$ and $d_g(Y,Z) = g(|b_1|,\dots,|b_n|)$ where $e^{ia_1},\dots,e^{ia_n}$ are the eigenvalues of XY^{-1}, and $e^{ib_1},\dots,e^{ib_n}$ are the eigenvalues of YZ^{-1}. Suppose XZ^{-1} has eigenvalues e^{ic_j}'s with $\pi \geq c_1 \geq \cdots \geq c_n > -\pi$. Then by Theorem 1, there exist Hermitian matrices $A, B, C = A + B$ with eigenvalues $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$ and $\tilde{c}_1 \geq \cdots \geq \tilde{c}_n$ such that if we replace $\tilde{c}_j$ by $\tilde{c}_j - 2\pi$ if $\tilde{c}_j > \pi$ and replace $\tilde{c}_j$ by $\tilde{c}_j + 2\pi$ if $\tilde{c}_j \leq -\pi$, then the resulting n entries will be the same as $c_1,\dots,c_n$ if they are arranged in descending order. Consequently, if $\|\mathbf{v}\|_k$ is the sum of the k largest entries of $\mathbf{v} \in \mathbb{R}^{1\times n}$ for $k = 1,\dots,n$, then

$$\begin{aligned}\|(|c_1|,\dots,|c_n|)\|_k &\leq \|(|\tilde{c}_1|,\dots,|\tilde{c}_n|)\|_k \\ &\leq \|(|a_1|,\dots,|a_n|)\|_k + \|(|b_1|,\dots,|b_n|)\|_k \\ &= \|(|a_1|+|b_1|,\dots,|a_n|+|b_n|)\|_k.\end{aligned} \tag{4}$$

Note that to arrive at the second inequality above, we have used the fact that

$$\|M+N\|_k \leq \|M\|_k + \|N\|_k \tag{5}$$

for any $n\times n$ complex-valued matrices M,N and for $k = 1,\dots,n$. Here $\|M\|_k$ is the Ky Fan k-norm, which is defined as the sum of the k largest singular values of M [11].

Since $g(\mathbf{u}) \leq g(\mathbf{v})$ for any $\mathbf{u},\mathbf{v} \in \mathbb{R}^{1\times n}$ if and only if $\|\mathbf{u}\|_k \leq \|\mathbf{v}\|_k$ for $k = 1,\dots,n$ [12, 13], it follows that

$$\begin{aligned}d_g(X,Z) &\leq g(|c_1|,\dots,|c_n|) \\ &\leq g(|a_1|+|b_1|,\dots,|a_n|+|b_n|) \\ &\leq g(|a_1|,\dots,|a_n|) + g(|b_1|,\dots,|b_n|) \\ &= d_g(X,Y) + d_g(Y,Z).\end{aligned} \tag{6}$$

Since the infimum in Eq. (3) is actually a minimum, there exist $r(X,Y), s(Y,Z) \in \mathbb{R}$ such that $d_g^{\triangledown}(X,Y) = d_g(e^{ir}X,Y)$ and $d_g^{\triangledown}(Y,Z) = d_g(e^{is}Y,Z) = d_g(Y,e^{-is}Z)$. From Eq. (6),

$$\begin{aligned}d_g^{\triangledown}(X,Y) + d_g^{\triangledown}(Y,Z) &= d_g(e^{ir}X,Y) + d_g(Y,e^{-is}Z) \\ &\geq d_g(e^{ir}X,e^{-is}Z) \\ &= d_g(e^{i(r+s)}X,Z) \\ &\geq d_g^{\triangledown}(X,Z).\end{aligned} \tag{7}$$

The proof is complete. $\square$

Example 3. For any $\mu = (\mu_1,\dots,\mu_n) \in \mathbb{R}^n$, define the μ-norm by

$$\|\mathbf{v}\|_\mu = \max\left\{\sum_{j=1}^n |\mu_j v_{i_j}| \colon \{i_1,\dots,i_n\} = \{1,\dots,n\}\right\}. \tag{8}$$

Clearly this is a family of symmetric norms; and the induced metrics and pseudo-metric on $U(n)$ are the families of metrics and pseudo-metrics introduced by Chau in Refs. [3, 4].

Example 4. One may pick g to be the ℓ_p norm defined by $\ell_p(\mathbf{v}) = \left(\sum_{j=1}^n |v_j|^p\right)^{1/p}$ for any $p \in [1,\infty]$. The induced metric on $U(n)$ has some interesting quantum information science meanings. In fact, it will be shown in Ref. [14] that this induced metric is a new family of indicator functions on the minimum resources needed to perform a unitary transformation. Moreover, these indicator functions are closely related to a new set of quantum speed limit bounds on time-independent Hamiltonians [14] generalizing the earlier results by Chau [3–5].

Remark 5. In the perturbation theory context, we consider $\tilde{X} = XE$, where E is very close to the identity. Suppose $X = e^{iA}$, where A has eigenvalues $\pi - \varepsilon > a_1 \geq$

$\cdots \geq a_n > -\pi+\varepsilon$, and $E = e^{iB}$ such that the eigenvalues of B lie in $[-\varepsilon, \varepsilon]$ for an $\varepsilon > 0$. Then we may conclude that $\tilde{X}$ has eigenvalues $\pi > c_1 \geq \cdots \geq c_n > -\pi$ such that $|c_j - a_j| \leq \varepsilon$.

III. SEVERAL INEQUALITIES ON PRODUCTS OF TWO UNITARY MATRICES

The proof technique used in Proposition 2 can be used to show an inequality generalizing a few similar ones originally reported by Chau in Ref. [3].

First, recall that given two non-increasing sequences of real numbers $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{u}' = (u'_1, \ldots, u'_n)$, we say that $\mathbf{u}$ is weakly sub-majorized by $\mathbf{u}'$ if $\sum_{j=1}^k u_j \leq \sum_{j=1}^k u'_j$ for $1 \leq k \leq n$. Furthermore, a real-valued function $h(\mathbf{u})$ is said to be Schur-convex if $h(\mathbf{u}) \leq h(\mathbf{u}')$ whenever $\mathbf{u}$ is weakly sub-majorized by $\mathbf{u}'$.

Proposition 6. Let

$$h(\mathrm{s}^{\downarrow}(A+B), \mathrm{s}^{\downarrow}(A), \mathrm{s}^{\downarrow}(B)) \leq 0 \tag{9}$$

be an inequality valid for all n-dimensional Hermitian matrices A and B, where $\mathrm{s}^{\downarrow}(A)$ denotes the sequence of singular values of A arranged in descending order. Suppose further that h is a Schur-convex function of its first argument whenever the second and third arguments are kept fixed. Then,

$$h(\mathrm{AAE}^{\downarrow}(XY), \mathrm{AAE}^{\downarrow}(X), \mathrm{AAE}^{\downarrow}(Y)) \leq 0 \tag{10}$$

where $\mathrm{AAE}^{\downarrow}(X)$ denotes the sequence of absolute value of the principal value of argument of the eigenvalues of an $n \times n$ unitary matrix X arranged in descending order. In other words, if the eigenvalues of the unitary matrix X are $e^{ia_1}, \ldots e^{ia_n}$ with $a_j \in (-\pi, \pi]$ for all j and $|a_1| \geq |a_2| \geq \cdots \geq |a_n|$, then $\mathrm{AAE}^{\downarrow}(X) = (|a_1|, |a_2|, \ldots, |a_n|)$.

Proof. Let $X, Y \in U(n)$. And write $X = \exp(iA)$, $Y = \exp(iB)$ and $XY = \exp(iC)$ where the eigenvalues of the Hermitian matrices A, B, C are all in the range $(-\pi, \pi]$. By Theorem 1, we can find a Hermitian matrix $\tilde{C}$ and $XY = \exp(i\tilde{C})$, where $\tilde{C} = W_1 A W_1^{-1} + W_2 B W_2^{-1}$ for some $W_1, W_2 \in U(n)$. Hence, $h(\mathrm{s}^{\downarrow}(\tilde{C}), \mathrm{s}^{\downarrow}(A), \mathrm{s}^{\downarrow}(B)) \leq 0$.

Note that the eigenvalues of $\tilde{C}$ need not lie on the interval $(-\pi, \pi]$. Yet, we can transform $\tilde{C}$ to C by replacing those eigenvalues a_j's of $\tilde{C}$ by $a_j + 2\pi$ if $a_j \leq -\pi$ and replacing them by $a_j - 2\pi$ if $a_j > \pi$. Obviously, $\mathrm{s}^{\downarrow}(C)$ is weakly sub-majorized by $\mathrm{s}^{\downarrow}(\tilde{C})$. Therefore,

$$\begin{aligned} &h(\mathrm{AAE}^{\downarrow}(XY), \mathrm{AAE}^{\downarrow}(X), \mathrm{AAE}^{\downarrow}(Y)) \\ =&h(\mathrm{s}^{\downarrow}(C), \mathrm{s}^{\downarrow}(A), \mathrm{s}^{\downarrow}(B)) \leq h(\mathrm{s}^{\downarrow}(\tilde{C}), \mathrm{s}^{\downarrow}(A), \mathrm{s}^{\downarrow}(B)) \leq 0. \end{aligned} \tag{11}$$

So, we are done. □

Corollary 7. Let $X, Y \in U(n)$ and that X, Y and XY have eigenvalues e^{ia_j}'s, e^{ib_j}'s and e^{ic_j}'s, respectively with $\pi \geq |a_1| \geq \cdots \geq |a_n| \geq 0$, $\pi \geq |b_1| \geq \cdots \geq |b_n| \geq 0$ and $\pi \geq |c_1| \geq \cdots \geq |c_n| \geq 0$. Then

$$\sum_{\ell=1}^{p} |c_{j_\ell + k_\ell - \ell}| \leq \sum_{\ell=1}^{p} (|a_{j_\ell}| + |b_{k_\ell}|), \tag{12}$$

for any $1 \leq j_1 < \cdots < j_p \leq n$ and $1 \leq k_1 < \cdots < k_p \leq n$ with $j_p + k_p - p \leq n$.

Proof. Eq. (12) is the direct consequences of Proposition 6 and the inequality

$$\sum_{\ell=1}^{p} \lambda^{\downarrow}_{j_\ell + k_\ell - \ell}(A+B) \leq \sum_{\ell=1}^{p} \left[\lambda^{\downarrow}_{j_\ell}(A) + \lambda^{\downarrow}_{k_\ell}(B)\right] \tag{13}$$

reported in Ref. [15]. Here $\lambda^{\downarrow}_j(A)$ denotes the jth eigenvalue of the Hermitian matrix A arranged in descending order. □

Remark 8. Actually, Eq. (13) belongs to a class of matrix inequalities in the form

$$\sum_{k \in \mathcal{K}} \lambda^{\downarrow}_k(A+B) \leq \sum_{i \in \mathcal{I}} \lambda^{\downarrow}_i(A) + \sum_{j \in \mathcal{J}} \lambda^{\downarrow}_j(B), \tag{14}$$

where A, B are $n \times n$ Hermitian matrices and $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are subsets of $\{1, 2, \ldots, n\}$ with equal cardinality. This class of matrix inequalities is sometimes called the Lidskii-type inequalities. Thus, Proposition 6 implies that every Lidskii-type inequality for Hermitian matrix induces a corresponding inequality for unitary matrix.

IV. RELATION TO THE HORN'S PROBLEM

In fact, Lidskii-type inequalities are closely related to the Horn's problem in matrix theory. Horn [16] conjectured that eigenvalues of the $n \times n$ Hermitian matrices A, B and $A+B$ are completely characterized by inequalities in the form Eq. (14) and the equality

$$\sum_{j=1}^{n} \lambda^{\downarrow}_j(A+B) = \sum_{j=1}^{n} \left[\lambda^{\downarrow}_j(A) + \lambda^{\downarrow}_j(B)\right]. \tag{15}$$

(That is to say, he believed that eigenvalues of A, B and $A+B$ obey Eq. (15) and certain Lidskii-type inequalities. Furthermore, given three decreasing sequences of real numbers $(a_j)_{j=1}^n$, $(b_j)_{j=1}^n$ and $(c_j)_{j=1}^n$ satisfying $\sum_{j=1}^n c_j = \sum_{j=1}^n (a_j + b_j)$ and the corresponding Lidskii-like inequalities in the form $\sum_{k \in \mathcal{K}} c_k \leq \sum_{i \in \mathcal{I}} a_i + \sum_{j \in \mathcal{J}} b_j$, then there exist Hermitian matrices A, B and $A+B$ whose eigenvalues equal a_j's, b_j's and c_j's, respectively.) Horn also wrote down a highly inefficient inductive algorithm to find the subsets $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ [16]. The Horn's problem was proven by combined works of

Klyashko [7] and Knutson and Tao [8]. Besides, the existence of a minimal set of Lidskii-type inequalities for the Horn's problem was also shown [7, 8, 17, 18]. In this regard, Remark 8 can be restated as follow: each of the minimal set of Lidskii-type inequalities for the Horn's problem induces an inequality for the eigenvalues of unitary matrices X, Y and XY.

Naturally, one asks if these corresponding inequalities completely characterizes the eigenvalues of the product of unitary matrices. This problem, which is sometimes called the multiplicative version of the Horn's problem, was solved by the combined works of Agnihorti and Woodward [10] and Belkale [17, 19] by means of quantum Schubert calculus. Phrased in the content of our current discussion, they proved the following. Let $e^{2\pi i\alpha_j}$'s, $e^{2\pi i\beta_j}$'s and $e^{2\pi i\gamma_j}$'s be eigenvalues of the $n \times n$ special unitary matrices X, Y and Z, respectively. Surely, one may constrain the phases of the eigenvalues by $\sum_{j=1}^{n} \alpha_j = 0$ and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > \alpha_1 - 1$. And β_j's and γ_j's are similarly constrained. Then, the eigenvalues of X, Y and Z satisfying $XYZ = I$ are completely characterized in the sense of the Horn's problem by inequalities in the form

$$\sum_{i\in\tilde{\mathcal{I}}} \alpha_i + \sum_{j\in\tilde{\mathcal{J}}} \beta_j + \sum_{k\in\tilde{\mathcal{K}}} \gamma_k \leq d \tag{16}$$

for some $d(\tilde{\mathcal{I}}, \tilde{\mathcal{J}}, \tilde{\mathcal{K}}) \in \mathbb{N}$ known as the Gromov-Witten invariant, where the subsets $\tilde{\mathcal{I}}$, $\tilde{\mathcal{J}}$ and $\tilde{\mathcal{K}}$ of $\{1, 2, \ldots, n\}$ are of the same cardinality. Similar to the Horn's problem, only a highly inefficient recursive algorithm is known to date to find these inequalities. Thus, it is instructive to see how to deduce our induced inequalities from those completely characterizing the multiplicative version of the Horn's problem as this problem seems to be non-trivial. In fact, a major difficulty of this approach is the different ways to order the eigenvalues e^{ia_j}'s — ours are ordered by the values of $|a_j|$'s while those arising from the multiplicative version of the Horn's problem are ordered by the values of a_j's. Note that in applications of matrix inequalities to practical problems such as numerical analysis and perturbation theory, it is often the case that one can deduce the useful results using the basic Lidskii-type inequalities in the form of Eqs. (12) or (13), and rarely would one use the full generalizations in Eq. (14). In fact, specializing the general results in (14) to deduce well known matrix inequalities may actually be quite involved. For example, see Theorem 3.4 and the discussion after it in Ref. [20]. In that paper, we obtained our main results using Thompson's theorem efficiently. As mentioned before, it will be instructive to use the general inequalities of the multiplicative version of Horn's problem to deduce our results, but it may not be easy and not very practical.

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